Upper/Lower Limits ($\limsup$ & $\liminf$)

Let (x_n) be a bounded seq, say by a and b: $a \le x_n \le b \ \forall n$

Let $T_n = \{x_m : m \ge n\} \ \forall n \in \mathbb{N}$. Define

$$s_n := \sup T_n \quad \text{and} \quad t_n := \inf T_n$$

Then $a \le t_n \le x_m \le s_n \le b \ \forall m \ge n$, and $t_n \uparrow_n$ and $s_n \downarrow_n$; consequently, by the BMCT, $t := \lim_n t_n$ and $s := \lim s_n = \inf\{s_n : n \in \mathbb{N}\}$

$= \sup\{t_n : n \in \mathbb{N}\}$

exist in $\mathbb{R}$. Note also that $t \le s$. Commonly t and s are respectively denoted by

$$\liminf_n x_n \quad \text{and} \quad \limsup_n x_n$$

that is,

$$\liminf_n x_n = \lim_n (\inf_{m \ge n} x_m) \ \& \ \limsup_n x_n = \lim_n (\sup_{m \ge n} x_m)$$

Th 1. Let notations as above for bounded seq (x_n) and let $\varepsilon > 0$.

(i) $\exists N_0 \in \mathbb{N}$ s.t. $x_m < \varepsilon + \limsup_n x_n \ \forall m \ge N_0$

(ii) $\exists$ infinitely many m such that $-\varepsilon + \limsup_n x_n < x_m$

(iii) $\exists$ infinitely many m such that

$$-\varepsilon + s < x_m < \varepsilon + s$$

Proof. Since $\lim_n s_n = s$, $\exists N_0 \in \mathbb{N}$ s.t.

$$s - \varepsilon < s_n < s + \varepsilon \quad \forall n \ge N_0.$$

$s_n = \sup\{x_m : m \ge n\}$

Students are then asked to check (i), (ii) and (iii).

Corollary 1. $\exists$ a subsequence (x_{n_k}) of (x_n) such that $\lim_k x_{n_k} = s$.

pf. Pick $n_1 \in \mathbb{N}$ such that

$$s-1 < x_{n_1} < s+1$$

By Th1 (iii) (applied to $\varepsilon = \frac{1}{2}$), take $n_2 > n_1$ such that

$$s-\frac{1}{2} < x_{n_2} < s+\frac{1}{2}$$

Inductively one can construct a strictly increasing sequence (n_k) of positive integers such that

$$s-\frac{1}{k} < x_{n_k} < s+\frac{1}{k} \quad \forall\, k \in \mathbb{N}$$

Then (x_{n_k}) is a subsequence of (x_n) and

$$s \le \lim_k x_{n_k} \le s \quad (\text{i.e. } \lim_k x_{n_k} = s) \quad \text{QED}$$

(Thus $s \in L$, where L is defined as below).

Th 2. Let $L \stackrel{\text{def}}{=} \{ l : l \text{ is a limit of a subsequence of } (x_n) \}$. Then $s = \max L$ (the largest ele of L).

proof. Let $l := \lim_k x_{m_k} \in L$ where (m_k) is a strictly increasing seq of natural numbers. Since $x_{m_k} \le s_{m_k}$ and $s = \lim_k s_{m_k} \left(= \lim_n s_n\right)$, it follows that $l \le s$.